Indian Statistical Institute
First Semester 2005-2006
Mid Semestral Exam
B.Math II Year
Differential Equations

Time: 3 hrs Date:12-09-05 [Max marks : 35]

Answer all the questions

1. Define a [continuous] function $g : R^2 \to R$ by $g(-x,\ -y) = -g(x,\ y)$,

$$\begin{aligned} g(x,y) &= y && \text{if} \quad 0 \le y \le \sqrt{x} \\ &= -y + 2\sqrt{x} && \text{if} \quad \sqrt{x} \le y \le 2\sqrt{x} \\ &= 0 && \text{otherwise} \end{aligned}$$

   Define $F : R \longrightarrow R$ by $F(x) = \int\limits_{-1}^{1} g(x,\ y)dy$

   Show that

   a) $F(x) = x$ for $|x| < 1/4$

   b) $F'(0) = 1$

   c) $\frac{\partial g}{\partial x}(0,\ y) = 0$ for $-1 < y < 1$

   d) $F'(0) \neq \int\limits_{-1}^{1} \frac{\partial g}{\partial x}(0, y)dy$ [5]

2. Let $g \in C_0^\infty(R)$, $g(0) = 1$ and g be real valued. Define $f(x,y) = g(x^2 + y^2)$. Show that one cannot find $u \in C^1(R^2)$ such that

$$\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right] u(x,y) = f(x,y)$$

   [4]

3. Define $P(t,\ x,y) = \frac{1}{\pi}\frac{y}{(x-t)^2+y^2}$ for $y > 0$ t, x in R.

   (a) Verify that $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] P(t,\ \ x,y) = 0$ for all t, x, y [1]

   b) Let $f : R \longrightarrow R$ be any bounded function and continuous at x_0. Define

$$u(x,y) = \int P(t,\ x,y)f(t)dt$$

   Show that $\lim\limits_{y \to 0} |u(x_0, y) - f(x_0)| = 0$ [2]

   c) Show that $\frac{\partial u}{\partial x}(x,y) = \int \frac{\partial}{\partial x} P(t,\ x,y)f(t)dt$ [4]

4. a) Let $f_1, f_2, \dots$ be a sequence of real valued C^1 functions on $[a,b]$. If $f_n \to g$ uniformly on $[a,b]$ and $f_n' \to h$ uniformly on $[a,b]$ show that g is differentiable and $g' = h$ [1]

   b) Let $f(x) = \sum\limits_1^\infty k^{-3} \sin(kx)$ show that f is differentiable on $[0, 2\pi]$

   [2]

5. Let $g : [0,\pi] \longrightarrow R$ be any C^1 function with $g(0) = g(\pi)$. Show that there exists real numbers $a_0, a_2, \dots$ with $\sum |a_k| < \infty$ and $g(x) = \sum\limits_0^\infty a_k \cos[kx]$ [4]

6. Find all solutions of

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

[3]

7. $a, h : R \to R$ are C^1 functions, $u : R^2 \longrightarrow R$ is a C^1 function satisfying $a(u(x,y))u_x + u_y = 0, \quad u(x,0) = h(x)$ Find u. [5]

8. a) If $u : R^2 \longrightarrow R$ is a C^2 function satisfying $u_{xx} - c^2 u_{yy} = 0$, [where c is a non zero constant]. Show that there exist functions f, g such that

$$u(x,y) = f(x+cy) + g(x-cy)$$

[2]

   b) Take $c = 1$ in (a). Let $PQRS$ be a rectangle in R^2 with sides parallel to the lines $x + y = 0$ and $x - y = 0$ show that

$$u(P) + u(R) = u(Q) + u(S)$$

[2]